ISOCHORIC FLOWS OF COMPLEXITY 2

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It is shown that for isochoric flows of complexity 2 for which only two first Rivlin–Ericksen tensors are different from zero, the so called viscometric constitutive model forms a general exact representation of the constitutive functional of an incompressible simple fluid.

Rivlin–Ericksen rheological models[1,2] of finite complexity N may be considered as partial representations of the constitutive functional of a simple material. According to the relations

$$\tau + P\mathbf{I} = \mathbf{f}^{N}(\mathscr{E}^{N}), \tag{1}$$

$$\mathscr{E}^{N} = (\mathbf{E}_1, \ldots, \mathbf{E}_N), \tag{2}$$

where $\mathbf{E}_k$ are Rivlin–Ericksen kinematic tensors, these models are *a priori* limited to the representation of mechanical behaviour of incompressible fluids, since neither the gradient of relative deformations nor the density appears in $\mathbf{f}^N$ as explicit arguments. Nevertheless, in accordance with the theory of simple materials we must take the deformation history $\mathbf{G}(s)$ as the primary kinematical argument.

Model (*1*) possesses then some meaning only if $\mathbf{G}(s)$ is N-times differentiable, *i.e.* all $\mathbf{E}_k$'s exist:

$$\mathbf{E}_k = (-1)^k \left(\frac{\mathrm{d}G(s)}{\mathrm{d}s}\right)\bigg|_{s=0}, \quad k = 1, \ldots, N, \tag{3}$$

$\mathbf{G}(s)$ is isochoric[2]:

$$\mathrm{Det}\,(\mathbf{I} + \mathbf{G}(s)) - \mathbf{I} = 0, \quad s \geqq 0, \tag{4}$$

$\mathbf{G}(s)$ is kinematically admissible, *i.e.* there exists a velocity field $\mathbf{v}(t, \mathbf{r})$ so that the following equation is satisfied[1] for $k = 1, \ldots, N$

$$\mathbf{E}_k = \frac{\mathrm{D}}{\mathrm{D}t}\mathbf{E}_{k-1} + \nabla\mathbf{v} \,.\, \mathbf{E}_{k-1} + \mathbf{E}_{k-1}\nabla\mathbf{v}^{\mathrm{T}}, \tag{5}$$

and

$$\mathbf{E}_0 = \mathbf{I}\,. \tag{6}$$

The stress response of an incompressible simple fluid may be represented by model (*1*) either exactly or in some approximation. This representation can be exact only if the course of $\mathbf{G}(s)$ is fully determined by parameter $\mathscr{E}^N$. This is satisfied especially for deformation histories of finite complexity[3]

$$\mathbf{G}(s) = \sum_{k=1}^{N} \frac{(-s)^k}{k!} \mathbf{E}_k\,, \tag{7}$$

$$\mathbf{E}_k = 0 \quad \text{for} \quad k > N\,. \tag{8}$$

All such $\mathscr{E}^N$'s which represent isochoric kinematically admissible deformation histories of complexity N will be denoted as the primary definition set $\mathscr{P}^N$ of complexity N. Parameters $\mathscr{E}^N \in \mathscr{P}^N$ must fulfil two independent sets of relations (*4*), (*5*) with supplementary conditions (*8*). Therefore it might be expected that set $\mathscr{P}^N$ is substantially more narrow than set $\mathscr{S}^N$ of all ordered N-component vectors of second order symmetrical tensors (in the three dimensional Euclidean space). This circumstance entails certain consequences for the algebraic representation of model (*1*) on the definition set $\mathscr{P}^N$.

Under an algebraic representation of an isotropic tensor function[2,4] we usually understand its unified form

$$\mathbf{f}^N(\mathscr{E}^N) = \sum_k a_k \mathbf{E}_1^{i_{k,1}} \cdot \ldots \cdot \mathbf{E}_N^{i_{k,N}}\,, \tag{9}$$

where a_k's are scalar functions of simultaneous scalar invariants of the argument $\mathscr{E}^N$. Algebraic representations of isotropic tensor functions on the definition set $\mathscr{S}^N$ were investigated thoroughly in[1,4]. If $\mathscr{P}^N$ is substantially more narrow than $\mathscr{S}^N$, it may be expected that the corresponding algebraic representations will be sharper.

In our work we investigate the algebraic representation of the Rivlin–Ericksen model of an incompressible fluid of complexity 2 on the definition set $\mathscr{P}^{(2)}$, where the stress response is represented exactly by this model. Its main contribution is in the proof of the identity of $\mathscr{P}^{(2)}$ with the category of viscometric deformation histories. The representation $\mathbf{f}^{(2)}(\mathscr{E}^2)$ for $\mathscr{E}^2 \in \mathscr{P}^{(2)}$ then only follows from known results of the rheodynamics of viscometric flows.

THEORETICAL

ISOCHORIC DEFORMATION HISTORIES OF COMPLEXITY 2

According to the definition[3], deformation histories of complexity $N = 2$ may be represented by a pair of kinematic tensors $\mathbf{E}_1, \mathbf{E}_2$. With respect to supplementary condition (*8*), the isochoricity conditions can be expressed by a set of six invariant algebraic relations $C_{k,N}$, $N = 2$, $k = 1, \ldots, 6$

$$C_{1,2}: \quad Z_1 = 0 \tag{9a}$$

$$C_{2,2}: \quad Z_{11} = Z_2 \tag{9b}$$

$$C_{3,2}: \quad Z_{111} = 3/2\, Z_{12} \tag{9c}$$

$$C_{4,2}: \quad Z_{112} = 1/4\,(Z_{22} + Z_2^2) \tag{9d}$$

$$C_{5,2}: \quad Z_{122} = Z_2 Z_{12} \tag{9e}$$

$$C_{6,2}: \quad 0 = 3Z_2 Z_{22} - Z_2^3 - 2Z_{222} \tag{9f}$$

with

$$Z_{i_1 \ldots i_p} = \operatorname{tr}\,(\mathbf{E}_{i_1} \cdot \ldots \cdot \mathbf{E}_{i_p})\,. \tag{10}$$

By the Cayley–Hamilton theorem, $C_{3,2}$ and $C_{6,2}$ can be rearranged to read

$$C_{3,2}: \quad \operatorname{Det}\,(\mathbf{E}_1) = 1/2\, Z_{12} \tag{11a}$$

$$C_{6,2}: \quad \operatorname{Det}\,(\mathbf{E}_2) = 0\,. \tag{11b}$$

The orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, in which the component matrix of $\mathbf{E}_2$ takes on the diagonal form, and the corresponding component evaluation of relevant tensor quantities $\nabla\mathbf{v}$, $\mathbf{E}_1$, $\mathbf{E}_2$ will be considered as canonical. It is obvious from (*11b*) that at least one of parameters p_i in the component form

$$m(\mathbf{E}_2) = m\Big(\sum_{i=1}^{3} p_i \mathbf{e}_i \otimes \mathbf{e}_i\Big) = \begin{pmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{pmatrix} \tag{12}$$

must be equal to zero. By rearranging the elements $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ in the canonical basis, the validity of the relations

$$p_3 = 0 , \quad p_1 \geqq p_2 \tag{13a,b}$$

can always be ensured. In this newly ordered canonical basis we will express the components of $\mathbf{E}_1$ as

$$m(\mathbf{E}_1) = \begin{pmatrix} w_1 & z_3 & z_2 \\ z_3 & w_2 & z_1 \\ z_2 & z_1 & w_3 \end{pmatrix} . \tag{14}$$

It is shown in Appendix *A* for $\mathbf{E}_1 \neq \mathbf{0}$, $\mathbf{E}_2 \neq \mathbf{0}$ that it is possible to represent $\mathbf{E}_1$, $\mathbf{E}_2$ by three independent real parameters q_1, q_2, ϕ in the form

$$p_1 = 2q_1^2 , \quad p_2 = 2q_2^2 , \tag{15a,b}$$

$$z_1 = q_2 \sin \phi , \quad z_2 = q_1 \cos \phi , \tag{16a,b}$$

$$z_3 = (q_1^2 \sin^2 \phi - q_2^2 \cos^2 \phi)/d , \tag{16c}$$

$$w_1 = -w_2 = w = 2q_1 q_2 \sin \phi \cos \phi/d , \tag{16d}$$

$$w_3 = 0 , \tag{16e}$$

where

$$d = \pm(q_1^2 \sin^2 \phi + q_2^2 \cos^2 \phi)^{1/2} \neq 0 . \tag{17}$$

For degenerate cases of $q_1^2 = q_2^2$ resp. $q_2 = 0$, the choice of the canonical basis according to *(12)*–*(13)* is not unambiguous, since $\mathbf{E}_2$ is in these cases invariant with respect to rotations of the reference system about the axis $\mathbf{e}_3$ resp. $\mathbf{e}_1$. As far as it is desirable, an unambiguous choice of the canonical basis can be achieved by additional conditions imposed on the component representation of $\mathbf{E}_1$, *e.g.* by

$$z_1 = 0 . \tag{18}$$

For $q_1 = q_2$, the canonical representation assumes the form

$$m(\mathbf{E}_2) = 2q_1^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad m(\mathbf{E}_1) = q_1 \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{19a,b}$$

and, similarly, for $q_2 = 0$

$$m(\mathbf{E}_2) = 2q_1^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad m(\mathbf{E}_1) = q_1 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{20a,b}$$

The latter case corresponds to viscometric deformation histories. It can be shown that the criterion of the viscometric kinematics within isochroic flows of complexity 2 is the zero value of any two of the canonical parameters p_1, p_2, p_3 or – in the invariant form – the zero value of the second principal invariant of $\mathbf{E}_2$:

$$\mathrm{II}(\mathbf{E}_2) = 1/2(\mathrm{tr}^2\, \mathbf{E}_2 - \mathrm{tr}\, \mathbf{E}_2^2) = 0\,. \tag{21}$$

KINEMATIC ADMISSIBILITY

According to (5), conditions of the kinematic admissibility for histories of complexity 2 may be expressed by the requirement of existence of such a field of rate gradients $\mathbf{L}$

$$\mathbf{L} = \nabla \mathbf{v}\,, \quad L_{ij} = \partial v_j / \partial x_i\,, \tag{22a,b}$$

that at a fixed material point the time functions $\mathbf{L} = \mathbf{L}(t)$, $\mathbf{E}_1 = \mathbf{E}_1(t)$, $\mathbf{E}_2 = \mathbf{E}_2(t)$, $\mathbf{E}_3(t) = \mathbf{0}$ satisfy the relations

$$\mathbf{E}_1 = \mathbf{L} + \mathbf{L}^{\mathrm{T}}\,, \tag{23}$$

$$\mathbf{E}_2 = \frac{\mathrm{D}}{\mathrm{D}t}\mathbf{E}_1 + \mathbf{L}\,.\,\mathbf{E}_1 + \mathbf{E}_1\,.\,\mathbf{L}^{\mathrm{T}}\,, \tag{24}$$

$$\mathbf{0} = \frac{\mathrm{D}}{\mathrm{D}t}\mathbf{E}_2 + \mathbf{L}\,.\,\mathbf{E}_2 + \mathbf{E}_2\,.\,\mathbf{L}^{\mathrm{T}}\,, \tag{25}$$

together with isochoric conditions (9).

System (23)–(25) has obviously a solution for such constant $\mathbf{L}(t) = \mathbf{L}_0$, for which it holds $\mathbf{L}_0^2 = \mathbf{0}$. These solutions correspond to viscometric deformation histories and in the canonical component representation they are equivalent to the relations $q_1 = \text{const.}$, $q_2 = 0$, $\phi = \text{const.}$ or $\phi = \phi(t)$, resp., for the rotation of the reference system about the axis $\mathbf{e}_1$ of the canonical basis. This solution is only a special case of results obtained by Noll[2] for the kinematics of the substantially stagnant motions. However, it does not follow from his results that there could not exist kinematically admissible isochoric motions of complexity 2 with time-dependent $\mathbf{L} = \mathbf{L}(t)$, $\mathbf{L}^2 \neq \mathbf{0}$. It is shown in Appendix *B* that system (23)–(25), (9) cannot have other solution than substantially stagnant, *i.e.* viscometric motion due to $\mathbf{L}_0^2 = \mathbf{0}$.

Thus we have shown that the category $\mathscr{P}^{(2)}$ of all kinematically admissible deformation histories which may be expressed in the form

$$\mathbf{G}(s) = (-s)\,\mathbf{E}_1 + \frac{s^2}{2}\,\mathbf{E}_2\,, \qquad (26)$$

and which are isochoric, *i.e.* admissible for incompressible materials, is identical with the category of viscometric deformation histories. The general stress response to these deformation histories can be represented by[5,6]

$$\mathbf{f}^{(2)}(\mathbf{E}_1, \mathbf{E}_2) = \eta \mathbf{E}_1 + \nu_1(\mathbf{E}_1^2 - 1/2\,\mathbf{E}_2) + \nu_2\,1/2\,\mathbf{E}_2 \qquad (27)$$

where η, ν_1, ν_2 are viscometric material functions of the single scalar argument $q = (1/2\,\mathrm{tr}\,\mathbf{E}_2)^{1/2}$.

DISCUSSION

The so called viscometric model[5,6] (27) is one of most general representations of the rheological constitutive functional, which can be at present explicitly constructed from experimental data. It is not clear whether this model can be applied outside its definition region $\mathscr{P}^{(2)}$. Without a detailed analysis, Pipkin and Tanner[6] recommend to use it as an "ad hoc approximation" for the rheological problems with the "nearly-viscometric" kinematics. Questions concerning the suitability of approximation (27) for kinematic situations, in which infinitesimal deformation perturbations are superimposed on the viscometric flow, are partially discussed in Pipkin's papers[7,8].

The use of model (27) as the constitutive approximation of "nearly-viscometric" kinematic situations has been criticized by Truesdell[9]. His critique may be considered as justified in point of nonexistence of a systematic theory which would specify quantitatively the character of the approximation if model (27) is applied to other than viscometric situations. However, the remaining part of this critique, which consists essentially of a proof that Pipkin and Tanner[6] had introduced the conception of the viscometric approximation of the general expansion of type (1) in an ambiguous and inconsistent manner, is based on contradictory standpoints. The relation $\mathrm{tr}\,\mathbf{E}_1^2 = \mathrm{tr}\,\mathbf{E}_2$ is exact not only for viscometric flows, but for all isochoric flows – that is for all kinematic situations for which expansions of type (1), in which the density of material does not appear explicitly, can possess some meaning. Consequently, if the argument q of viscometric material functions is defined by the relation $q = (1/2\,\mathrm{tr}\,\mathbf{E}_2)^{1/2}$, it is consistent with the definition $q = (1/2\,\mathrm{tr}\,\mathbf{E}_1^2)^{1/2}$, independent of higher order kinematic tensors $\mathbf{E}_3, \mathbf{E}_4, \ldots$, and, eventually, unambiguous.

It must be of course stressed that it does not follow from our considerations that set $\mathscr{P}^{(2)}$ would have to include all kinematically admissible isochoric deformation histories of a finite resp. countable complexity which are unambiguously determined by actual values of the first two kinematic tensors $(\mathbf{E}_1, \mathbf{E}_2)$.

APPENDIX

A. Consequences of Isochoricity Conditions

In the canonical component representation, set (*9*) of algebraic equations may be written as

$$C_{1,2}: \quad w_1 + w_2 + w_3 = 0\,, \tag{28}$$

$$C_{2,2}: \quad w_1^2 + w_2^2 + w_3^2 + 2(z_1^2 + z_2^2 + z_3^2) = p_1 + p_2\,, \tag{29}$$

$$C_{3,2}: \quad \begin{vmatrix} w_1 & z_3 & z_2 \\ z_3 & w_2 & z_1 \\ z_2 & z_1 & w_3 \end{vmatrix} = 1/2(w_1p_1 + w_2p_2)\,, \tag{30}$$

$$C_{4,2}: \quad (w_1^2 + z_3^2 + z_2^2)\,p_1 + (z_3^2 + w_2^2 + z_1^2)\,p_2 = 1/2(p_1^2 + p_2^2 + p_1p_2)\,, \tag{31}$$

$$C_{5,2}: \quad w_1p_1^2 + w_2p_2^2 = (p_1 + p_2)(w_1p_1 + w_2p_2)\,, \tag{32}$$

$$C_{6,2}: \quad (0 = p_1p_2p_3\,, \quad p_3 = 0)\,. \tag{33}$$

It is obvious from $C_{2,2}$ that this set possesses a solution in real parameters w_i, z_i, p_i only if $(p_1 + p_2) \geqq 0$, whereas for $p_1 + p_2 = 0$ it holds $w_i = z_i = p_i = 0$.

Then it is sufficient to set $p_1 + p_2 > 0$ or, for the sake of unambiguousness, $p_1 \geqq p_2, p_1 > 0$. By (*32*), generally it holds

$$(w_1 + w_2)\,p_1 \cdot p_2 = 0\,, \tag{34}$$

hence either $w_1 + w_2 = 0$ or $p_2 = 0$.

For $p_2 = 0$ and on the assumption of $p_1 \neq 0$, we arrive at the following set by linear combination of relations (*28*)–(*31*)

$$z_1^2 + w_2(w_1 + w_2) = 0\,, \tag{35a}$$

$$z_2^2 + z_3^2 + w_1^2 = 1/2p_1\,, \tag{35b}$$

$$-z_2^2w_2 + 2z_1z_2z_3 + z_3^2(w_1 + w_2) = 1/2p_1w_1\,. \tag{35c}$$

We will show that

$$w_1 + w_2 \neq 0 \Rightarrow w_1 = 0 \,\&\, w_2 = 0\,, \tag{36}$$

which leads necessarily to $w_1 + w_2 = 0$ also for $p_2 = 0$. We will assume that $p_2 = 0$ and consider the three following possible cases:

a) Let $w_1 = 0$. According to (*35a*), $z_1^2 + w_2^2 = 0 \Rightarrow w_2 = 0$.

b) Let $w_2 = 0$. According to (*35a*), $z_1 = 0$, and by (*35c*) it holds $z_3^2 w_1 = \frac{1}{2} p_1 w_1$. Let us start with $w_1 \neq 0$. Then $z_3^2 = \frac{1}{2} p_1$ and by (*35b*) $w_1 = 0$, which is in contradiction. For $w_1 = 0$, solutions of case *b*) and *a*) are identical, *i.e.*

$$z_1 = w_2 = w_1 = p_2 = 0\,, \tag{37a}$$

$$z_2^2 + z_3^2 = 1/2 p_1 > 0\,. \tag{37b}$$

c) Finally let $w_1 \neq 0$, $w_2 \neq 0$, $w_1 + w_2 \neq 0$. By inserting (*35b*) into (*35c*) we obtain

$$(z_1 z_2 + w_2 z_3)^2 = (w_2 w_1)\, w_1^2\,, \tag{38}$$

which is in contradiction with (*35a*) since

$$-w_2 w_1 - w_2^2 > 0 \quad \text{and} \quad w_2 w_1 > 0$$

cannot hold simultaneously. Thus, neither case *c*) admits a real solution for $w_1 + w_2 \neq 0$.

It remains to show that for $w_1 + w_2 = 0$, *i.e.*

$$w_1 = -w_2 \equiv w\,, \tag{39a}$$

$$w_3 = 0\,, \tag{39b}$$

set (*9*) possesses real solutions only if $p_2 \geqq 0$. Relations (*29*)–(*31*), which – on inclusion of results from (*39a,b*) – are not satisfied identically, may be now expressed in the form

$$z_1^2 + z_2^2 + z_3^2 + w^2 = 1/2(p_1 + p_2)\,, \tag{40}$$

$$2 z_1 z_2 z_3 = w[(p_1 - p_2)/2 + (z_1^2 - z_2^2)]\,, \tag{41}$$

$$z_1^2 p_1 + z_2^2 p_2 = 1/2 p_1 p_2\,. \tag{42}$$

By introducing the nonnegative auxiliary parameter X^2 through the relation

$$2 z_2^2 = X^2 p_1 \geqq 0\,, \tag{43a}$$

by (*42*) we have

$$2 z_1^2 = (1 - X^2)\, p_2 \geqq 0\,. \tag{43b}$$

According to (*40*) it holds $2(z_3^2 + w^2) = p_2 + (1 - X^2)(p_1 - p_2) \geqq 0$,

$$(1 - X^2)(p_1 - p_2) \geqq -p_2\,. \tag{43c}$$

For $p_2 < 0$ we obtain $(1 - X^2) . p_2 \leqq 0$ according to (43c), which is for $X^2 \neq 1$ in contradiction with requirement (43b). For $X^2 = 1$ relations (43a) and (40) lead to $p_2 \geqq 0$.

Since $0 \leqq X^2 \leqq 1$, $p_1 \geqq p_2 \geqq 0$, we can introduce the notation

$$X = \cos \phi\,, \quad \sin \phi = \pm(1 - X^2)^{1/2}\,, \tag{44a,b}$$

$$p_1 = 2q_1^2\,, \quad p_2 = 2q_2^2 \tag{45a,b}$$

for arbitrary real ϕ, q_1, q_2, $q_1^2 \geqq q_2^2$.

It remains to examine the region of admissible values of real q_1, q_2, ϕ. In terms of parameters (q_1, q_2, ϕ), relation (42) is satisfied identically and (40), (41) may be rewritten as a set for z_3, w:

$$z_3^2 + w^2 = d^2\,, \tag{46}$$

with

$$d = \pm(q_2^2 \cos^2 \phi + q_1^2 \sin^2 \phi)^{1/2}\,, \tag{47}$$

and

$$(2q_1q_2 \cos \phi \sin \phi)\, z_3 = (q_1^2 \sin^2 \phi - q_2^2 \cos^2 \phi)\, w\,. \tag{48}$$

If $d = 0$, obviously it holds $E_1 = E_2 = 0$. For $d \neq 0$, set (46), (47) possesses a general solution represented by relations (15a,b) in the main text. This solution includes also all cases when some of parameters q_2, $\sin \phi$, $\cos \phi$, resp. q_1 are equal to zero even if $d \neq 0$.

B. Consequence of Conditions of Kinematic Admissibility

Let $L_{ij} = \partial v_j/\partial x_i$, $L_{ij} = L_{ij}(t)$ be Cartesian components of the velocity gradient at the given material point and expressed with respect to the canonical basis $\mathbf{e}_1(t)$, $\mathbf{e}_2(t)$, $\mathbf{e}_3(t)$. Spin components for even permutations (i, j, k) of indices (1, 2, 3) will be denoted by the relations

$$\omega_k = L_{ij} - L_{ji}\,. \tag{49}$$

From relations (23) and (39), the following combinations of L_{ij} may be found:

$$z_k = L_{ij} + L_{ji} \quad (i \neq j)\,, \tag{50}$$

$$1/2\, w = L_{11} = -L_{22}\,, \quad 0 = L_{33}\,. \tag{51}$$

Relations (24) can be expressed in the component form

$$2q_1^2 = \dot{w} + w^2 + (z_3 + \omega_3)\, z_3 + (z_2 - \omega_2)\, z_2\,, \tag{52}$$

$$2q_2^2 = -\dot{w} + w^2 + (z_3 - \omega_3)\, z_3 + (z_1 + \omega_1)\, z_1\,, \tag{53}$$

$$0 = (z_2 + \omega_2)\, z_2 + (z_1 - \omega_1)\, z_1\,, \tag{54}$$

$$0 = 2\dot{z}_3 - 2w\omega_3 + (z_2 - \omega_2)\, z_1 + (z_1 + \omega_1)\, z_2\,, \tag{55}$$

$$0 = 2\dot{z}_2 + w(2z_2 + \omega_2) + (z_3 + \omega_3) z_1 + (z_1 - \omega_1) z_3 , \quad (56)$$

$$0 = 2\dot{z}_1 - w(2z_1 - \omega_1) + (z_3 - \omega) z_2 + (z_2 + \omega_2) z_3 , \quad (57)$$

with $\dot{w} = Dw/Dt$, *etc.*

The component form of relations (*25*) reads as

$$0 = \frac{D}{Dt} q_1^2 + wq_1^2 , \quad 0 = \frac{D}{Dt} q_2^2 - wq_2^2 \quad (58), (59)$$

$$0 = (z_2 + \omega_2) q_1^2 , \quad 0 = (z_1 - \omega_1) q_2^2 \quad (60), (61)$$

$$0 = (z_3 - \omega_3) q_1^2 + (z_3 + \omega_3) q_2^2 . \quad (62)$$

This set will be considered as the formulation of the initial value problem with undetermined initial conditions for $q_1 = q_1(t)$, $q_2 = q_2(t)$, $\phi = \phi(t)$, $\omega_i = \omega_i(t)$, since the remaining parameters may be expressed through q_1, q_2, ϕ according to (*15*), (*16*). It follows from (*58*), (*59*) that q_1, q_2 are not mutually independent functions, but that it holds necessarily

$$q_1(t) . q_2(t) = K = \text{const} . \quad (63)$$

Let us start with $q_2 = 0$. We will show that in this case it must hold $q_1 = $ const. Above all, relation (*16*) yields $w = z_1 = 0$ and (*55*), (*56*) may be rearranged to read

$$2 \frac{D}{Dt} z_3 = \omega_1 z_2 , \quad 2 \frac{D}{Dt} z_2 = -\omega_1 z_3 . \quad (64a,b)$$

Let us consider (*64a,b*) and the consequence of (*16*) in the form of $z_3^2 + z_2^2 = q_1^2$. Either it holds $z_2 = 0 \Rightarrow z_3 = q_1 = $ const., or $z_3 = 0 \Rightarrow z_2 = q_1 = $ const., or $\omega_1 = 0 \Rightarrow (z_3 = $ const., $z_2 = $ const.$) \Rightarrow q_1 = $ const., or finally for $z_2 \neq 0$, $z_1 \neq 0$ by combining (*64a,b*)

$$\frac{D}{Dt} (z_3^2 + z_2^2) = 0 \Rightarrow q_1^2 = \text{const} . \quad (64c)$$

The case of $q_2 = 0$, $q_1 = $ const. corresponds to a viscometric kinematics with a time-independent shear rate q_1 at the given material point.

Now we will show that set (*52*)—(*62*) does not have any solution for $q_2 \neq 0$, $q_1 \neq 0$.

By (*60*)—(*62*) we have

$$z_2 + \omega_2 = 0 , \quad z_1 - \omega_1 = 0 , \quad z_3 - \omega_3 (q_1^2 - q_2^2)/(q_1^2 + q_2^2) = 0 . \quad (65a-c)$$

Let us begin with $q_1 \neq 0$, $q_2 \neq 0$, $w = 0$, then by (*58*), (*59*) it holds $q_1 = $ const., $q_2 = $ const., and by (*41*) $z_1 = 0$, $z_2 = 0$, $z_3 = 0$, which is possible only if $\phi(t) = $ const., $z_1 = $ const., $z_2 = $ const., $z_3 = $ const. In this case relation (*55*) takes the form of $0 = z_1 z_2$. Firstly let $z_2 \neq 0$, then $z_1 = 0$ and, by *(57)*, $z_3 - \omega_3 = 0$. This is in contradiction with *(53)*, according to which it holds $0 \neq 2q_2^2 = (z_3 - \omega_3) z_3 + 2z_1^2$. A similar contradiction is obtained from the assumption $z_1 \neq 0$.

Let us further assume that $q_1 \neq 0$, $q_2 \neq 0$, $w \neq 0$, $z_3 = 0$. By *(65c)* it holds either $\omega_3 = 0$ or $q_1^2 = q_2^2$. Firstly let $\omega_3 = 0$, then *(55)* and *(65)* lead to $z_1 z_2 = 0$ and, by *(41)*, $w = 0$ which is in contradiction. Let then $q_1^2 = q_2^2 \neq 0$, thus *(63)* yields $q_1 = q_2 =$ const. and by *(58)*, *(59)* we have $w = 0$, which is again in contradiction.

Finally let us assume that $w \neq 0$, $z_3 \neq 0$, $q_1 \neq 0$, $q_2 \neq 0$. By combining *(52)*, *(53)*, *(55)* and *(65)* we can arrive at the relation

$$\frac{D}{Dt}\left(z_3^2 + w^2\right) = 0\,, \quad i.e.\ z_3^2 + w^2 = M^2 = \text{const}\,. \qquad (66a,b)$$

For $w \neq 0$, the time-independent variable may be replaced by the independent variable q_1, since by *(58)* it holds

$$\frac{D}{Dt}(\ ) = -1/2wq_1 \frac{D}{Dq_1}(\ )\,. \qquad (67)$$

All the variables w, q_2, z_i may be expressed by *(15)*, *(16)*, *(63)*, *(66)* as functions of two constant parameters K, M and the variable q_1, which enables to reduce set *(52)*–*(57)* to a set of algebraic equations for K, M, q_1. For $q_1 \neq 0$, this set possesses a solution only for $K = 0$, *i.e.* for $q_2 = 0$, which is in contradiction with the original assumption.

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